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A Stochastic Model of Bin-Packing

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Recent research in combinatorial bin-packing models is extended to a stochastic model in which an arbitrary distribution of piece sizes is assumed. The asymptotic expected bin occupancy is obtained for a simple on-line algorithm. Convergence properties are also presented so that, for a given set of pieces, this measure can be related to the expected number of bins required relative to an optimization rule.

1. INTRODUCTION

The past few years have seen a mounting interest in the analysis of combinatorial models of bin-packing problems (Johnson (1974), Johnson *et al.* (1974), and Coffman (1978)). In the classical problem one seeks to minimize the number of equal capacity bins needed for the packing of a given collection of pieces. There are a great many applications of this problem in Computer Science and, indeed, throughout industry. In particular, stock cutting is a wide ranging application that includes cutting variable size pieces or segments from standard sheets of paper in the printing industry, from standard cloth measures in the textile industry, from standard stock in the building industry, and so on. In Computer Science important storage allocation problems appear as bin-packing problems; these include packing records into auxiliary storage and word lay-out problems. The interested reader is referred to Johnson *et al.* (1974) for further discussion.

Our interest focuses on a probability model. As usual, there are two complementary properties of this approach: The greater demands of the probabilistic analysis, in particular our inability to tackle any inherently non-regenerative process, limit the study to the simpler, on-line packing algorithms; on the other hand, probability distributions of performance are more informative than the worst-case results of combinatorial analyses. We shall concentrate on the so-called Next-Fit algorithm and develop expected values for the comparative performance of this rule and an optimization rule.

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Little has been published to date on the stochastic characterization of packing processes. The only analysis that we know of, which is related to the problem at hand, is due to Shapiro (1977). There, certain assumptions are introduced (e.g., infinite support of the piece size distribution and a "memoryless" property for truncated exponential random variables), on which an analysis leading to approximate results is based. For pieces with mean size not too large when compared with the bin size (ratios of up to $\frac{1}{3}$ are considered), the calculations of expected waste and number of bins used produce quite good approximations.

In the next section we introduce the basic model and describe the fundamental properties of the processes of interest. In Section 3 convergence properties of the packing process are analyzed and in Section 4 the results are specialized to the case of a uniform distribution of piece sizes. Performance measures and concluding remarks are provided in the final section.

2. The One-Dimensional Model

We assume an infinite sequence of bins $\langle B_i, i \geqslant 1\rangle$ whose common capacity is taken, without loss of generality, to be 1. The pieces to be packed are specified in an infinite sequence $S = \langle X_i, i \geqslant 1\rangle$. The X_i denote both piece names and piece sizes; no harm will come from this ambiguity. The piece sizes (or lengths) are assumed to be independent random variables with the common distribution function $G(x)$ defined on the unit interval $[0, 1]$ (X will denote a generic piece-size random variable). Consistent with our purposes in this paper, $G(x)$ will usually be assumed to possess a density, and no atoms.

According to the Next-Fit algorithm the bins are packed in the sequence $B_1, B_2, \ldots$. First, pieces are drawn in sequence from S and placed in B_1 until a piece, say X, is encountered which will not fit into the remaining unused capacity of B_1. At that point, starting with X, B_2 is packed in an identical manner; the first piece not fitting in B_2 commences B_3, whereupon the process repeats. Figure 1 illustrates this packing algorithm.

Let L_i denote the *level* of B_i, that is the sum of the piece-sizes in B_i, once B_{i+1} has been started. The process $\{L_i, i \geqslant 1\}$ will concern us for the rest of the paper. We observe that given the value of L_i, the probability distribution for

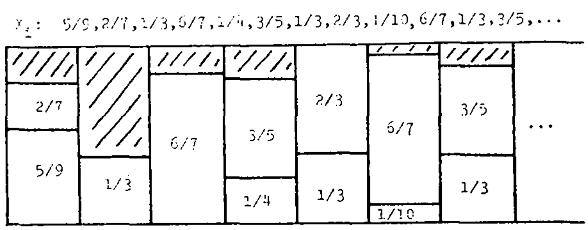

Fig. 1. Next-fit packing.

L_{i+1} is completely determined and independent of the values assumed by L_j, $j < i$. Thus, $\{L_i\}$ is a Markov chain with the continuous state space $[0, 1]$, where the value of L_i will be conveniently termed the state of the process. The stationary, one-step transition probability is denoted $K(x, y) = \Pr\{L_{i+1} \leqslant y \mid L_i = x\}$.

Other, related processes which will be of interest are $\{W_i, N_i, i \geqslant 1\}$, which are respectively the size of the first piece packed in B_i, and the number of pieces stored in B_i. It is surprising to note that while $\{W_i\}$ is a Markov chain just as $\{L_i\}$ is, $\{N_i\}$ is not (i.e., $\Pr\{N_{i+2} \mid N_i, N_{i+1}\} \neq \Pr\{N_{i+2} \mid N_{i+1}\}$.) However, since the distribution of N_i is determined uniquely by W_i (see the discussion following Theorem 3 below), this is of little consequence.

The chain $\{L_i\}$ is characterized by the relation

$$F_{L_{i+1}}(y) = \int_0^1 K(x, y)\, dF_{L_i}(x), \qquad i \geqslant 1 \tag{1}$$

where $F_{L_1}(y) = K(1, y)$ by definition of the Next-Fit algorithm. In the context of (1) $K(x, y)$ is also called a kernel.

For the calculation of $K(x, y)$ the following notation is convenient. Let S_n denote the sum of $n \geqslant 0$ independent, identically distributed piece-sizes, where S_0 takes on the value zero only. By definition of the Next-Fit rule we must have $W_{i+1} > 1 - L_i$, $i \geqslant 1$. Hence $K(x, y) = 0$ for all $y \leqslant 1 - x$. Now $L_{i+1} \leqslant y$ if and only if for some $n \geqslant 0$, the sum of W_{i+1} and the next n piece sizes is no greater than y, but the sum of W_{i+1} and the next $n + 1$ piece-sizes exceeds 1. Hence, using the Markov property of $\{L_i\}$ and a complete set of events,

$$K(x, y) = \sum_{n=0}^{\infty} \Pr\{W_{i+1} + S_n \leqslant y,\ W_{i+1} + S_n + X > 1 \mid L_i = x\}. \tag{2}$$

The completeness of this family of events assures the stochasticity of the kernel (i.e., $K(x, 1) = 1$). The rest of this section prepares the ground for a proof of its regularity.

The conditional distribution of W_{i+1} given that $L_i = x$ is simply

$$F_{W_{i+1}|L_i}(w \mid x) = [G(w) - G(1 - x)]/[1 - G(1 - x)],$$

for $1 - x < w$, and 0 otherwise. Thus (2) can be expressed as

$$K(x, y) = \sum_{n=0}^{\infty} \int_{1-x}^{y} \Pr\{S_n \leqslant y - w,\ X > 1 - w - S_n\} \frac{dG(w)}{1 - G(1 - x)}.$$

Finally, therefore, using the independence of successive X_i, we get

$$\begin{aligned} K(x, y) &= \sum_{n=0}^{\infty} \int_{1-x}^{y} \int_0^{y-w} \frac{1 - G(1 - w - s)}{1 - G(1 - x)}\, dF_{S_n}(s)\, dG(w), \qquad y > 1 - x \\ &= 0, \qquad y \leqslant 1 - x. \end{aligned} \tag{3}$$

Later, using a uniform distribution for $G(x)$, we shall work out closed form results.

Note that in the limit $x \to 0$, $K(x, y)$ degenerates to a distribution concentrated at $y = 1$. Also, although $K(x, y)$ for $x > 0$ will generally be a continuous function of y in our application, this is not required for all that follows, and in particular, it will not possess a continuous density.

Now suppose $G(x)$ has a density $g(x)$ that is strictly positive on $[0, 1]$; i.e., $(a, b) \subset [0, 1]$ and $b > a$ imply that $G(b) - G(a) > 0$. Under this assumption, which we shall weaken later, a number of qualitative properties of the chain $\{L_i\}$ can be established.

1. For each x, $K(x, y)$ must clearly be a monotonically increasing function of y in the interval $[1 - x, 1]$. Moreover, the two-step transition probability $K^{(2)}(x, y) = \Pr\{L_{i+2} \leqslant y \mid L_i = x\}$ increases monotonically with y over the entire interval $[0, 1]$. To see this it is sufficient to observe that for all x and z in $(0, 1]$ and for all $y > \max\{1 - x, 1 - z\}$, $K(x, y) > 0$ and $K(y, z) > 0$; i.e., every point z in B_{i+2} is "reachable" by L_{i+2} from every point x in B_i. Inductively, this also applies to the n-step transition probability $K^{(n)}(x, y)$, $n \geqslant 2$, where

$$F_{L_{i+n}}(y) = \int_0^1 K^{(n)}(x, y)\, dF_i(x) \tag{4}$$

2. Since $L_i \leqslant \frac{1}{2}$ implies $W_{i-1} > \frac{1}{2}$ and hence $L_{i+1} > \frac{1}{2}$, we have $E[L_{i+1} \mid L_i \leqslant \frac{1}{2}] > \frac{1}{2}$. Moreover, for each x there exists an $\frac{1}{2} < \alpha < 1$ such that for all $x > \alpha$ we have $E[L_{i+1} \mid L_i = x] < x$. Indeed, the only requirement for this property is that $G(x)$ not be concentrated at a single point. Thus, the process has a tendency to move from points near the boundaries towards the interior. Also, since $L_i + L_{i+1} > 1$ for all $i \geqslant 1$, it is easy to see that $E[L_i] > \frac{1}{2}$ for all $i \geqslant 1$.

It follows directly from the above properties (see Tweedie (1975), for example) that $\{L_i\}$ is an ergodic Markov chain having a strictly positive stationary probability distribution $F_L(y) = \lim_{i\to\infty} F_{L_i}(y)$ satisfying

$$F_L(y) = \int_0^1 K(x, y)\, dF_L(x) = \lim_{n\to\infty} K^{(n)}(x, y) \tag{5}$$

An even stronger property of $\{L_i\}$ can be verified; viz. the convergence to $F_L(y)$ is geometrically fast. This is shown in the next section.

3. Convergence Properties

We say that $F_{L_n}(y)$ converges geometrically (or exponentially) fast to $F(y)$ if there exist positive constants a and b such that for all n sufficiently large

$|K^{(n)}(x, y) - F(y)| \leqslant a\epsilon^{-bn}$ for all x in $[0, 1]$. It is well-known (see Loève (1963), for example) that if the so-called Markov measure

$$\Delta_n = \sup_{x,y} \sup_{z} \Delta_n(x, y, z); \qquad \Delta_n(x, y, z) = | K^{(n)}(x, z) - K^{(n)}(y, z)| \quad (6)$$

is such that $\Delta_n < 1$ for some $n_1 \geqslant 1$, then the convergence of $F_{L_i}(y)$ to $F(y)$ is exponential with a rate no slower than $\Delta_{n_1}^{1/n_1}$. The following result therefore shows exponential convergence for our particular chain.

THEOREM 1. *For $\{L_i\}$ we have $\Delta_1 = 1$, but for all $h \geqslant 2$, $\Delta_h < 1$.*

Proof. For Δ_1 consider (6) with $x = 1$, $y = 0$ and $z = 1$. Using (3) we find that the first term in (6) is 1 (the maximum of a distribution) and the second term is 0. Hence, $\Delta_1 = 1$.

For $h \geqslant 2$ we note that $\Delta_h(x, y, z)$ is a bounded function; and since $K^{(h)}(x, 0) = 0$, $K^{(h)}(x, 1) = 1$ for all $x \in [0, 1]$ we have $\Delta_h(x, y, 1) = \Delta_h(x, y, 0) = 0$ for all $x, y \in [0, 1]$. Thus, considered as a function of z, $\Delta_h(x, y, z)$ will, for given x and y, attain its supremum over $z \in (0, 1)$. But as argued in Section 2 $K^{(h)}(x, z)$ is strictly positive over $z \in [0, 1]$ and hence $K^{(h)}(x, z) < 1$ for $z < 1$ and for all x. It follows immediately that $\Delta_h < 1$, $h \geqslant 2$. ∎

Remark. A similar result would also hold if $G(x)$ did not have a density, but then $K^{(h)}(\cdot)$ would not be strictly positive and the proof would require modification.

Using this result we can characterize the expected efficiency of Next-Fit packings, relative to the best achievable. To this end we consider first the expected cumulative size of the pieces packed in the prefix $B_1, \ldots, B_m$, for arbitrary m. The following result shows that this differs by at most a constant from the value obtained when approximating $E(L_i)$ by $\bar{L} = \lim_{i\to\infty} E(L_i)$.

THEOREM 2. *There exists a constant γ such that for all m*

$$\left| m\bar{L} - \sum_{i=1}^{m} E[L_i] \right| < \gamma \qquad (8)$$

Proof. We begin by writing

$$\sum_{i=1}^{m} E[L_i] = m\bar{L} + \sum_{i=1}^{m} (E[L_i] - \bar{L})$$

Therefore,

$$\left| m\bar{L} - \sum_{i=1}^{m} E[L_i] \right| \leqslant \sum_{i=1}^{m} | E[L_i] - \bar{L} | \qquad (9)$$

We complete the proof by showing that the sum $\gamma^* = \sum_{i=1}^{\infty} | E[L_i] - \bar{L} |$ exists.

Since the L_i are non-negative random variables we have

$$E[L_i] = 1 - \int_0^1 F_{L_i}(y)\, dy.$$

Consequently, γ^* can be expressed as

$$\gamma^* = \sum_{i=1}^{\infty} \left| \int_0^1 [F_{L_i}(y) - F_L(y)]\, dy \right|$$

from which, using (4), we derive

$$\gamma^* \leqslant \sum_{i=1}^{\infty} \int_0^1 \int_0^1 | K^{(i-1)}(x, y) - F_L(y)|\, dF_1(x)\, dy \tag{10}$$

where $K^{(0)}(x, y)$ may be written using the Heaviside theta function as $\theta(y - x)$. From the definition of Δ_h is not difficult to verify (see Loève (1963) for example) for some fixed $h \geqslant 1$

$$| K^{(n)}(x, y) - F_L(y)| \leqslant \Delta_h^{\lfloor n/h \rfloor}$$

Substituting into (10) we have

$$\gamma^* \leqslant \sum_{i=0}^{\infty} \Delta_h^{\lfloor i/h \rfloor} \tag{11}$$

Convergence of this sum for any $h \geqslant 2$ follows directly from Theorem 1. ∎

Remark. Extension of the results to the more general case where $g(x)$ is strictly positive with support $[0, a]$, $a \leqslant 1$, is easily managed. Note that in this case the state space reduces to $[1 - a, 1]$, so that the earlier characterizations of $\{L_i\}$ must be specialized accordingly. Indeed, it is only this sort of specialization which would be anticipated for any more general $G(x)$ that might arise in applications. For the case of support on $[0, a]$ modifications to (3) are restricted to limits of integration and the initial value of n in the summation, which must become $\lfloor 1/a \rfloor$ (all bins must have at least $\lfloor 1/a \rfloor$ pieces). In contrast to the case $a > \frac{1}{2}$ note that for $a \leqslant \frac{1}{2}$, $K(x, y)$ and its density $k(x, y)$ can be continuous over the entire state space $[1 - a, 1]$. Furthermore, the convergence properties, and in particular Theorem 2 do not depend on the everywhere-positivity of $g(x)$ or even on its existence; they would continue to hold so long as $G(\cdot)$ is not concentrated in one point.

4. Results for Piece Sizes Uniformly Distributed

Important, specific instances of our problem occur when the piece sizes are assumed to be uniformly distributed over the interval $[0, a]$, $a \leqslant 1$. Although numerical solutions for general a can be worked out in principle, closed-form expressions for measures of interest are not generally possible. Such expressions are available for the case $a = 1$, however, which we shall now develop. It is simple to argue that smaller values of a will yield very similar results, except that convergence may be even faster.

THEOREM 3. *For piece sizes uniformly distributed over* $[0, 1]$ *we have the kernel*

$$\begin{aligned} K(x, y) &= 1 - (1 - y)\, e^{-(1-y)} \left(\frac{e^x}{x}\right); \qquad 1 - x < y \leqslant 1 \\ &= 0; \qquad 0 \leqslant y \leqslant 1 - x \end{aligned} \tag{12}$$

with the invariant measure (*stationary distribution*) *and expectation*

$$F_L(y) = y^3; \qquad 0 \leqslant y \leqslant 1, \qquad L = \tfrac{3}{4}. \tag{13}$$

Proof. First, from $G(x) = x$, $0 \leqslant x \leqslant 1$, we find

$$F_{S_n}(s) = \frac{s^n}{n!}, \qquad f_{S_n}(s) = \frac{s^{n-1}}{(n-1)!}; \qquad 0 \leqslant s \leqslant 1$$

Note that $\sum_{n=1}^{\infty} f_{S_n}(s) = e^s$. Thus, exploiting the uniform convergence in (3) we move the summation within the integrals, separate out the $n = 0$ term and obtain

$$K(x, y) = 1 - \frac{x}{2} + \int_{1-x}^{1} \int_{0}^{y-w} \frac{w + s}{x}\, e^s \, ds \, dw$$

The expression in (12) follows upon integration.

Turning now to the stationary distribution we substitute into (5) and get

$$F_L(y) = \int_{1-y}^{1} \left[1 - (1 - y)\, e^{-(1-y)} \left(\frac{e^x}{x}\right)\right] dF_L(x)$$

Assuming $F_L(y)$ has the density $f_L(y)$, differentiation yields

$$f_L(y) = y e^{-(1-y)} \int_{1-y}^{1} \frac{e^x}{x} f_L(x)\, dx$$

Defining $g(y) = f_L(y)/(ye^y)$ we have

$$g(y) = \int_{1-y}^{1} e^{-(1-2x)} g(x)\, dx \tag{14}$$

This homogeneous, linear integral equation has the solution $g(y) = cye^{-y}$. (Perhaps the most straightforward way to obtain it is by differentiating (14) twice, obtaining $g''(y) + 2g'(y) + g(y) = 0$, and then evaluating (14) and its first derivative at $y = 0$.) Since $f_L(y)$ is a density we must have $c = 3$, $f_L(y) = 3y^2$ and the stationary distribution in (13). ∎

From Theorem 3 simple results yield the distributions of the first piece size W in a "limiting" bin, the number of pieces in such a bin, and their mean values.

COROLLARY 1. *For uniformly distributed piece sizes*

$$f_W(w) = \tfrac{3}{2}(2w - w^2), \qquad 0 \leqslant w \leqslant 1$$
$$P_n = 3(n^2 + 3n + 1)/(n + 3)!, \qquad n \geqslant 1$$

where $P_n = \lim_{i\to\infty} \Pr\{|B_i| = n\}$. *Computing mean values we find* $E(W) = \frac{5}{8}$ *and* $E(N) = \frac{3}{2}$.

Proof. The *pdf* $f_W(w)$ is obtained directly from the statement following (2). The *pmf* P_n follows from

$$P_{k+1} = \int_{w=0}^{1} f_W(w) \int_{s=0}^{1-w} \Pr\{S_k = s, X > 1 - w - s\}\, ds\, dw$$

upon substituting $f_W(\cdot)$. The expectations are found in the usual way. ∎

Quantifying the parameters in Theorem 2 we have

THEOREM 4. *For uniformly distributed piece sizes*

$$m \leqslant \frac{4}{3} \sum_{i=1}^{m} E[L_i] + 3 \tag{15}$$

Proof. First, the expected value of the stationary distribution in (13) is given by $\bar{L} = \frac{3}{4}$. To verify the additive constant we calculate Δ_2 as defined in (6). This requires determining $K^{(2)}(x, z)$ which, on one iteration of (1), can be found from

$$K^{(2)}(x, z) = \int_0^1 K(y, z)\, K(x, dy)$$

Making use of (12) we obtain

$$K^{(2)}(x, z) = \int_{\max\{1-x,1-z\}}^{1} ye^{-(1-y)}\left(\frac{e^x}{x}\right)\left[1 - (1 - z)\,e^{-(1-z)}\left(\frac{e^y}{y}\right)\right] dy$$

and hence

$$K^{(2)}(x, z) = \begin{cases} K'(x, z) = \dfrac{e^x}{x}\left[ze^{-z} - \dfrac{1-z}{2}(e^z - e^{-z})\right] & z < x \\ K''(x, z) = 1 - \dfrac{1-z}{2}\,e^z\left(\dfrac{e^x - e^{-x}}{x}\right) & z \geqslant x \end{cases}$$

Based on the structure of $K^{(2)}(x, z)$ the following case analysis is convenient.

1. $z \geqslant \max\{x, y\}$. Omitting routine details we find

$$\max_z\{\max_{x\leqslant z} K''(x, z) - \min_{y\leqslant z} K''(x, z)\} < 2/10$$

2. $z < \min\{x, y\}$. For this case

$$\max_z\{\max_{x>z} K'(x, z) - \min_{y>z} K'(x, z)\} < 3/10$$

3. $x > z \geqslant y$. We obtain

$$\max_z\{\max_{x>z} K'(x, z) - \min_{z\geqslant y} K''(x, z)\}$$
$$< \max_z\{\max_{z\geqslant y} K''(x, z) - \min_{x>z} K'(x, z)\} < 3/10$$

Thus, we have $\Delta_2 < 3/10$. Next, from (9) and (11) we may choose in Theorem 2, $h = 2$ and any $\gamma \geqslant \sum_{n=0}^{\infty} \Delta_2^{\lfloor n/2 \rfloor} = 2/(1 - \Delta_2)$. Since $\Delta_2 < 3/10$, $\gamma = 3$ suffices. ∎

5. Discussion

We begin by comparing the performance of the NF algorithm against that of an optimum packing. Theorem 4 gives rise to a simple approach, avoiding difficulties in perhaps more direct approaches.

Consider the pieces NF would pack into B_1 to B_m. An optimization algorithm would certainly need no less than $\sum_{i=1}^{m} L_i$ bins. Thus, the expected number, OPT_m, of bins required to optimally pack these sequences satisfies $OPT_m \geqslant \sum_{i=1}^{m} E(L_i) > m\bar{L} - \gamma\bar{L}$, or $OPT_m/m \sim \bar{L}$, for large m.

A more "direct" comparison is to define $NF(n)$ and $OPT(n)$ as the expected number of bins required to pack a list of n pieces using the corresponding algorithms. While $\{N_i\}$ does not constitute a Markov chain, it is simply related to one. In fact, for uniformly distributed piece sizes it is easy to derive the probability generating function of N_{i+1}, given that $L_i = x : \eta(z \mid x) = E(z^{N_{i+1}} \mid L_i = x) = 1 + (e^{xz} - 1)(1 - 1/z)/x$. Thus, $\{N_i\}$ enjoys geometrical convergence, in distribution, just as $\{L_i\}$ does. Note that when we pack a finite sequence, only the contents of the *last* bin used are affected by the finiteness. From Theorem 2 and the above results for the relation between $\{N_i\}$ and $\{L_i\}$ we infer the existence of a constant δ such that $| m\bar{N} - \sum_{i=1}^{m} E(N_i) | < \delta$, for all m. Fixing the length of the sequence (i.e., $\sum N_i = n$), the last inequality can be rewritten as $| NF(n) \cdot \bar{N} - n | < \delta$. Then, seeing that for any (finite) sequence S, $OPT(S) \geqslant \sum_{i=1}^{NF(S)} L_i$, the law of large numbers (which holds for recurrent Markov chains) allows us to write the right hand side, for large $| S |$, as $\bar{L} \cdot NF(| S |)$; hence, writing n for $| S |$, we have $OPT(n) \sim \bar{L} \cdot NF(n)$, for large n.

Considering the values of n for which this relationship may be indicative, the following table is informative. (It was computed for $X \sim U(0, 1)$.) Note the extremely fast convergence of the mean values. For piece sizes with support smaller than the size of the bin the convergence will be even faster.

i	1	2	3	4	5
$E(L_i)$	.71828	.75797	.74846	.75035	.749935
$E(N_i)$	1.71828	1.47624	1.5064	1.49881	1.50027

In concluding this presentation we should point out that NF is one of the simpler possible packing procedures; and even here, though qualitative results were not hard to derive, obtaining numerical indicators is well nigh impossible except for the very simplest of distributions. Even a slight sophistication of the packing procedure enormously exacerbates the calculation, since the dependence between successive operations is tightened.

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